Discrete Mathematics: Combinatorics and Graph Theory

Practice Exam 3

Instructions. Solve any 5 questions and state which 5 you would like graded. Note that this is a sample exam, and while it bears some similarity to the real exam, the two are not isomorphic.

1. Solve the following recurrence relations:

   (a) $a_n = 6a_{n-1} - 9a_{n-2}$ when $a_0 = 2, a_1 = 21$.
   Factor the characteristic polynomial $x^2 - 6x + 9 = 0 \Rightarrow (x-3)(x-3) = 0 \Rightarrow x = -3$. Then $a_n = \alpha(3)^n + \beta \cdot n \cdot (3)^n$. Applying initial conditions $a_0 = 2 = \alpha(3)^0 + \beta \cdot 0 \cdot (3)^0 \Rightarrow \alpha = 2$. $\alpha_1 = 21 = \alpha(3)^1 + \beta \cdot 1 \cdot (3)^1 \Rightarrow 21 = 6 + 3\beta \Rightarrow \beta = 5$. The final solution is $a_n = 2(3)^n + 5 \cdot n \cdot (3)^n$.

   (b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with $a_0 = 3, a_1 = 6$, and $a_2 = 0$.
   $x^3 - 2x^2 - x + 2 = 0 \Rightarrow x^2(x-2) - 1(x-2) = 0 \Rightarrow (x-1)(x+1)(x-2) = 0 \Rightarrow x = 1, x = -1, x = 2$. The solution is of the form $a_n = \alpha_1(1)^n + \alpha_2(-1)^n + \alpha_3(2)^n$. Applying the initial conditions and solving the resulting system: $\alpha_1 = -2, \alpha_2 = 6, \alpha_3 = -3$. Therefore the solution can be expressed $a_n = 2(-1)^n + 6(1)^n + (-1)(2)^n$.

   (c) $a_n = 2a_{n-1} + 1$ when $a_1 = 1$.
   $a_2 = 2 \cdot 1 + 1 = 3\,, a_3 = 2 \cdot 3 + 1 = 7\,, a_4 = 2 \cdot 7 + 1 = 15$. Each value is twice the previous minus one: $a_n = 2 \cdot 2^n - 1$.

   (d) $na_n = (n-2)a_{n-1} + 2$ when $a_1 = 1$.
   Recall that $\sum_{i=1}^{n} i = n(n-1)/2 \Rightarrow n(n-1) = 2 \cdot \sum_{i=1}^{n} i$. Multiply both sides by $(n-1)$:

$$\begin{aligned}
n(n-1)a_n &= (n-1)(n-2)a_{n-1} + 2(n-1) \\
&= (n-2)(n-3)a_{n-2} + 2(n-2) + 2(n-1) \\
&= (n-3)(n-4)a_{n-3} + 2(n-3) + 2(n-2) + 2(n-1) \\
&= 2(1 + \cdots + n - 1) \\
&= 2 \cdot \sum_{i=1}^{n} i \\
n(n-1)a_n &= n(n-1) \\
a_n &= 1
\end{aligned}$$

   (e) $a_n = 3a_{n-1} + 10a_{n-2} + 7.5^n$ where $a_0 = 4$ and $a_1 = 3$.
   The characteristic equation of the associated homogeneous relation is $x^2 - 3x - 10 = 0 \Rightarrow (x-5)(x-2) = 0 \Rightarrow x = -5\,, x = -2$. Therefore $a_n^h = \alpha(5)^n + \beta(-2)^n$. Since $f(n) = 7.5^n$ is of the form cx^n consider Anx^n so that $a_n^{(p)} = Anx^n = An5^n$. Plugging into the recurrence relation $An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7.5^n$. Dividing by 5^{n-2} yields $An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7.5^2 \Rightarrow 25An = 15An - 15A + 10An - 20A + 175 \Rightarrow 35A = 175 \Rightarrow A = 5$. Therefore $a_n^{(p)} = An5^n = 5n5^b = n5^{n+1}$. The solution of the recurrence relation can be written as $a_n = a_n^{(h)} + a_n^{(p)} = \alpha(5)^n + \beta(-2)^n + n5^{n+1}$. Applying initial conditions and solving gives $\alpha = -2, \beta = 6$ for $a_n = -2 \cdot (5)^n + 6 \cdot (-2)^n + n5^{n+1}$.

2. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

   (a) exactly three boys?

$$P(\text{exactly 3 boys}) = \binom{5}{3}(.51)^3(.49)^2$$

(b) at least one boy?

$$P(\text{at least 1 boy}) = 1 - P(\text{no boys}) = 1 - \binom{5}{0}(.51)^0(.49)^5$$

(c) at least one girl?

$$P(\text{at least 1 girl}) = 1 - P(\text{no girls}) = 1 - \binom{5}{5}(.51)^5(.49)^0$$

(d) all children of the same sex?

$$\binom{5}{5}(.51)^5(.49)^0 + \binom{5}{0}(.51)^0(.49)^5$$

3. How many members of the set $S = \{1, 2, 3, \cdots, 105\}$ have nontrivial factors in common with 105? Hint: use the inclusion-exclusion principle.
105 has a prime factorization $105 = 3 \times 5 \times 7$, so elements in S will have common factors with 105 if they are divisible by 3, 5 or 7. Define A as elements of S divisible by 3, B as elements of S divisible by 5 and C as elements divisible by 7. There are 35 numbers from 1 to 105 divisible by 3 so the subset A contains 35 elements. Similarly there are 21 elements in the subset B and 15 elements in the subset C. Consider $A \cap B$, the subset of elements divisible by both 3 and 5. There are seven numbers between 1 and 105 divisible by 15 (we had worked this out in checking that our formula for the recurrence in question 1.2 was correct), therefore $|A \cap B| = 7$. Similarly $A \cap C$ is the subset of elements divisible by both 3 and 7 (21) and $B \cap C$ is the subset of elements divisible by both 5 and 7 (35). Therefore $|A \cap C| = 5$ and $|B \cap C| = 3$. The only number divisible by 105 is 105 so $|A \cap B \cap C| = 1$. Applying the inclusion-exclusion principle:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 35 + 21 + 15 - 7 - 5 - 3 + 1 \\ &= 57 \end{aligned}$$

4. Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that
Let H denote HIV and P denote positive.

(a) a patient testing positive for HIV with this test is infected with it?

$$P(H|P) = \frac{P(P|H)P(H)}{P(P|H)P(H) + P(P|\bar{H})P(\bar{H})} = \frac{0.98 \times 0.08}{0.98 \times 0.08 + 0.03 \times 0.92} = 0.740$$

(b) a patient testing positive for HIV with this test is not infected with it?

$$P(\bar{H}|P) = 1 - P(H|P) = 1 = 0.74 = 0.26$$

(c) a patient testing negative for HIV with this test is infected with it?

$$P(H|\bar{P}) = \frac{P(\bar{P}|H)P(H)}{P(\bar{P}|H)P(H) + P(\bar{P}|\bar{H})P(\bar{H})} = \frac{0.02 \times 0.08}{0.02 \times 0.08 + 0.97 \times 0.92} = 0.002$$

(d) a patient testing negative for HIV with this test is not infected with it?

$$P(\bar{H}|\bar{P}) = 1 - P(H|\bar{P}) = 1 - 0.002 = 0.998$$

5. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two or three stairs at a time. What are the initial conditions? How many ways can this person climb a flight of eight stairs?

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

The initial conditions are $a_0 = a_1 = 1$, $a_2 = 2$ or $a_1 = 1, a_2 = 2$ and $a_3 = 4$.

$$\begin{aligned}
a_3 &= a_0 + a_1 + a_2 = 1 + 1 + 2 = 4 \\
a_4 &= a_1 + a_2 + a_3 = 1 + 2 + 4 = 7 \\
a_5 &= a_2 + a_3 + a_4 = 2 + 4 + 7 = 13 \\
a_6 &= a_3 + a_4 + a_5 = 4 + 7 + 13 = 24 \\
a_7 &= a_4 + a_5 + a_6 = 7 + 13 + 24 = 44 \\
a_8 &= a_5 + a_6 + a_7 = 13 + 24 + 44 = 81
\end{aligned}$$

6. Show that $\mathbb{E}_Y(\mathbb{E}_X(P(X|Y)) = \mathbb{E}_X(X)$

$$\begin{aligned}
\mathbb{E}_Y(\mathbb{E}_X P(X|Y)) &= \sum_y \mathbb{E}_X(P(X|Y = y)P(Y = y) \\
&= \sum_y \sum_x x \cdot P(X = x|Y = y)P(Y = y)
\end{aligned}$$

Recall that by Bayes rule $P(A|B)P(B) = P(B|A)P(A)$

$$\begin{aligned}
&= \sum_x \sum_y x \cdot P(Y = y|X = x)P(X = x) \\
&= \sum_x x \cdot P(X = x) \sum_y P(Y = y|X = x) \\
&= \sum_x x \cdot P(X = x) \\
&= \mathbb{E}_X(X)
\end{aligned}$$

7. Consider a random walk (a drunk stumbling in one dimension) with step sizes of S_i where S_i is $+1$ with probability p and -2 with probability $q = 1 - p$. Let $T_n = \sum_{i=‘}^{m} S_i$ be the displacement after a fixed (not random) number of steps n. Find the probability $P(T_n = t)$ and the mean and variance of T_n in terms of n and p.
Denote the number of $+1$ moves with n steps as X_n. We can then define the number of $+2$ moves is defined as $n - k$ when k denotes the number of $+1$ moves. The total displacement is then $t = k + 2(n - k) \Rightarrow k = 2n - t$.

   (a) We want $P(T_n = t)$ which follows a Binomial distribution.

   $$P(T_n = t) = P(X_n = 2n - t) = \binom{n}{2n - t} p^{2n-t}(1 - p)^{t-n}$$

   (b) $\mathbb{E}(S_i) = p + -2(1 - p) = 3p - 2$. Each step is iid so that $\mathbb{E}(T_n) = n\mathbb{E}(S_i) = 3np - 2n$

   (c) $\mathrm{Var}(S_i) = \mathbb{E}(S_i^2) - \mathbb{E}(S_i)^2 = p + (1 - p)(-2)^2 - (3p - 2)^2 = 9p(1 - p)$. Each step is iid so that $\mathrm{Var}(T_n) = n\mathrm{Var}(S_i) = 9np(1 - p)$